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Uniform Sampling of the Infinite Noncooperative Game on Unit Hypercube and Reshaping Ultimately Multidimensional Matrices of Player's Payoff Values

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Abstract **– The paper suggests a method of obtaining an approximate solution of the infinite noncooperative game on the unit hypercube. The method is based on sampling uniformly the players' payoff functions with the constant step along each of the hypercube dimensions. The author states the conditions for a sufficiently accurate sampling and suggests the method of reshaping the multidimensional matrix of the player's payoff values, being the former player's payoff function before its sampling, into a matrix with minimally possible number of dimensions, where also maintenance of one-to-one indexing has been provided. Requirements for finite NE-strategy from NE (Nash equilibrium) solution of the finite game as the initial infinite game approximation are given as definitions of the approximate solution consistency. The approximate solution consistency ensures its relative independence upon the sampling step within its minimal neighborhood or the minimally decreased sampling step. The ultimate reshaping of multidimensional matrices of players' payoff values to the minimal number of dimensions, being equal to the number of players, stimulates shortened computations.**

Keywords **– Systems, man, and cybernetics; Decision theory; Computational efficiency; Mathematical model.**

I. Noncooperative Game Models

Mathematically, a game is a means for rational resources allocation to meet the growing demands and requirements. And many events, systems and processes, related to the inequality between resources and pretensions, are modeled with noncooperative games [1], [2]. Particularly, these games involve a few players in military process and jurisprudence [3], [4]. Also, there may be more players (up to a few tens) in economical and social-ecological gaming models [5], [6]. Sometimes a two-sided noncooperative game is the most appropriate model for removing uncertainties in technical problems [6], [7]. For instance, this is for preventing a denial of service, when a server (reservoir) runs out of resources while a number of queries (demands) is not less than the rejection number [8], [9].

Naturally, that far from any noncooperative game solution in NE (Nash equilibrium) strategies ensures equilibrium, utility, and fairness [2], [10], [11]. However, NE-solutions render a lot of the refined or modified principles of optimality, allowing to smooth differences in utility and equity [2], [10], [11]. Mainly, they are principles of Pareto equilibrium [2], [6], [8], [10], [13], [14], Mertens-stable equilibrium [15], trembling hand perfect equilibrium [16], proper equilibrium [17], [18], correlated equilibrium [19], sequential equilibrium [20], [21], quasi-perfect equilibrium [18], [22], [23], perfect Bayesian equilibrium [18], [20], [24], [25], quantal response equilibrium [26], [27], self-confirming equilibrium [28], [29], strong Nash equilibrium [30], [31], Markov perfect equilibrium [32], [33]. The question is only to find NE-solutions as fast as possible.

II. Solving Noncooperative Games on Compact Action Spaces

Finding NE-solutions in even the finite noncooperative game is a computational difficulty [8], [34], [35]. Locally, solving dyadic games with three players takes some technique of visualization of the cube of situations in pure strategies [6], [10], whereupon dyadic games with four players and more are solved purely in analytics, requiring more computational resources [10], [36]. Naturally, that finite noncooperative games with greater numbers of pure strategies at their players (three and more) are significantly hard to solve them [10], [37], [38]. Moreover, often an admissible player's action is described with a series of its continuous parameters, constituting thus an infinite (continuous) set of pure strategies [1], [6], [7], [12], [39], [40]. If this continuous set is compact then it is easy to find an isomorphic game to the initial one, that the set of every player's pure strategies would be Euclidean finite-dimensional subspace [6], [10], [12], [23], [41]. Normally, the spoken subspace may be a unit cube of the appropriate dimension [6], [10], [11]. Nonetheless, compact games, having solutions at least in mixed strategies for measurable payoff functions [1], [6], [10], [11], [41], [42], cannot be solved by a universal algorithmic approach, unless they are finite games.

III. Tasks for the Goal Attainment

Clearly, a proper conversion of the infinite noncooperative game on unit hypercube into a finite game lets to have a guaranteed NE-solution [6], [10] to the conflict object. Therefore, the goal of this paper is to state the conditions or requirements of that conversion. To simplify the finite game more there will be reconfigured sets of players' pure strategies, letting to get rid of dimensionalities and to have the single dimension for each player. For the goal attainment there are tasks to state the following:

1. Conditions for sampling the players' payoff functions correctly, being sufficiently accurate for practice experience.
2. Method of reshaping the multidimensional matrix of the player's payoff values (being the former player's payoff function before its sampling) into a matrix with minimally

possible number of dimensions. Maintenance of one-to-one indexing should be provided.

3. Requirements on that any finite NE-strategy (from NE-solution of the finite game as the approximation of the initial infinite game) must not be too dependent upon the sampling step. Or, in other words, NE-strategy support must be independent upon the sampling step within some tolerable dependence.

IV. Sampling of Players' Payoff Functions

Consider a noncooperative game

$$\left\langle \left\{U_n\right\}_{n=1}^{N}, \left\{K_n\left(\mathbf{X}\right)\right\}_{n=1}^{N} \right\rangle \tag{1}$$

of $N \in \mathbb{N} \setminus \{1\}$ players, in which the n th player acts within the unit hypercube U_n of M_n -dimensional points (pure strategies)

$$\mathbf{X}_n = \left[x_{nm}\right]_{1\times M_n} \in \bigtimes_{m=1}^{M_n} [0;\,1] = U_n \subset \mathbb{R}^{M_n}$$

$$\text{by } M_n \in \mathbb{N} \quad \forall\, n = \overline{1,\,N}\,, \tag{2}$$

and in the situation

$$\mathbf{X} = \left\{\mathbf{X}_n\right\}_{n=1}^{N} \in \bigtimes_{n=1}^{N} U_n = \bigtimes_{k=1}^{\sum_{i=1}^{N} M_i} [0;\,1] \subset \mathbb{R}^{\sum_{i=1}^{N} M_i} \tag{3}$$

gets the payoff $K_n\left(\mathbf{X}\right)$. Each of the measurable payoff functions $\left\{K_n\left(\mathbf{X}\right)\right\}_{n=1}^{N}$ is defined on the unit $\left(\sum_{i=1}^{N} M_i\right)$ – dimensional hypercube

$$U = \bigtimes_{n=1}^{N} U_n = \bigtimes_{k=1}^{\sum_{i=1}^{N} M_i} [0;\,1] \subset \mathbb{R}^{\sum_{i=1}^{N} M_i} \tag{4}$$

and it is supposed that they all are differentiable with respect to any of variables $\left\{\left\{x_{nm}\right\}_{m=1}^{M}\right\}_{n=1}^{N}$. Also there exist mixed derivatives of each of functions $\left\{K_n\left(\mathbf{X}\right)\right\}_{n=1}^{N}$ by any combination of variables $\left\{\left\{x_{nm}\right\}_{m=1}^{M}\right\}_{n=1}^{N}$ in any situation (3), where every variable is included no more than just once.

While sampling uniformly, let S be the number of intervals between the selected points in each of the dimensions of every hypercube from $\left\{U_n\right\}_{n=1}^{N}$. Remember that each dimension is the unit segment now. Tolerating the utmost case of sampling, where endpoints of the unit segment are included into the sampling necessarily, $S \in \mathbb{N}$. The constant sampling step is S^{-1}. Thus in *m*th dimension *n*th player instead of the segment $[0;\,1]$ of values of *m*th component of its pure strategy $\mathbf{X}_n$ in (2) now possesses the set of points

$$D_m^{\langle n\rangle}\left(S\right) = \left\{x_{nm}^{\langle s\rangle}\right\}_{s=1}^{S+1}$$

$$\text{by } x_{nm}^{\langle s\rangle} = \frac{s-1}{S} \quad \forall\, m = \overline{1,\,M_n} \text{ and } \forall\, n = \overline{1,\,N}\,. \tag{5}$$

The choice of number S should not annul specificities of the players' payoff functions. These specificities consist in local extremums and gradient over hypersurfaces $\left\{K_n\left(\mathbf{X}\right)\right\}_{n=1}^{N}$. That is $\forall\, s = \overline{1,\,S}$ there ought to be

$$\frac{\partial^{\sum_{i=1}^{N} M_i} K_n\left(\mathbf{X}\right)}{\partial x_{11}\partial x_{12}\ldots\partial x_{1M_1}\partial x_{21}\partial x_{22}\ldots\partial x_{2M_2}\ldots\partial x_{N1}\partial x_{N2}\ldots\partial x_{NM_N}} \geqslant 0 \text{ or}$$

$$\frac{\partial^{\sum_{i=1}^{N} M_i} K_n\left(\mathbf{X}\right)}{\partial x_{11}\partial x_{12}\ldots\partial x_{1M_1}\partial x_{21}\partial x_{22}\ldots\partial x_{2M_2}\ldots\partial x_{N1}\partial x_{N2}\ldots\partial x_{NM_N}} \leqslant 0$$

$$\forall\, x_{nm} \in \left[x_{nm}^{\langle s\rangle};\, x_{nm}^{\langle s+1\rangle}\right],\ \forall\, m = \overline{1,\,M_n}\,,\ \forall\, n = \overline{1,\,N}\,, \tag{6}$$

and

$$\left|\frac{\partial^{\sum_{i=1}^{N} M_i} K_n\left(\mathbf{X}\right)}{\partial x_{11}\partial x_{12}\ldots\partial x_{1M_1}\partial x_{21}\partial x_{22}\ldots\partial x_{2M_2}\ldots\partial x_{N1}\partial x_{N2}\ldots\partial x_{NM_N}}\right| \leqslant \alpha$$

$$\forall\, x_{nm} \in \left[x_{nm}^{\langle s\rangle};\, x_{nm}^{\langle s+1\rangle}\right],\ \forall\, m = \overline{1,\,M_n}\,,\ \forall\, n = \overline{1,\,N} \tag{7}$$

for some $\alpha > 0$, implying tolerable fluctuations of the players' payoff functions on every of segments

$$\left\{\left\{\left\{\left[x_{nm}^{\langle s\rangle};\, x_{nm}^{\langle s+1\rangle}\right]\right\}_{s=1}^{S}\right\}_{m=1}^{M_n}\right\}_{n=1}^{N}.$$

But here it is clear that conditions (6) are satisfied only if there are no extremums on any of intervals

$$\left\{\left\{\left\{\left(x_{nm}^{\langle s\rangle};\, x_{nm}^{\langle s+1\rangle}\right)\right\}_{s=1}^{S}\right\}_{m=1}^{M_n}\right\}_{n=1}^{N}.$$

Obviously, the players' payoff functions, whose extremums can be reached at points, having only components $\left\{\left\{\left\{x_{nm}^{\langle s\rangle}\right\}_{s=2}^{S}\right\}_{m=1}^{M_n}\right\}_{m=1}^{N}$, can be defined just artificially, rather than be expected to occur for real conflict processes. Hence, the real conditions for sampling the players' payoff functions correctly remain in inequalities (7). The conditions (6) may be optionally checked to underscore the accurateness of the sampling (but they will be unlikely satisfied, unless there are two players and minimum of dimensions).

Parameter α is pre-assigned on some practical reasoning. Considering the value

$$v_\alpha = \max_{n\in\left\{\overline{1,\,N}\right\}} \max_{\mathbf{X}\in U} K_n\left(\mathbf{X}\right) - \min_{n\in\left\{\overline{1,\,N}\right\}} \min_{\mathbf{X}\in U} K_n\left(\mathbf{X}\right),$$

it can be said that $\alpha \leqslant 0.01\cdot v_\alpha$ or $\alpha \leqslant 0.001\cdot v_\alpha$, are sufficiently accurate for practice experience. Nevertheless, the parameter α may be taken lesser to have the game approximate solution consistent enough, what is going to be spoken about further.

V. Reshaping of Multidimensional Matrices of Players' Payoff Values

After having sampled the hypersurfaces $\left\{K_n\left(\mathbf{X}\right)\right\}_{n=1}^{N}$ on the unit hypercube (4) the *n*th player gets $\left(\sum_{i=1}^{N} M_i\right)$-dimensional matrix $\mathbf{P}_n = \left[p_J^{\langle n \rangle}\right]_{\mathsf{F}}$ of the format $\mathsf{F} = \bigtimes_{k=1}^{\sum_{i=1}^{N} M_i} \left(S+1\right)$, whose $\left(\sum_{i=1}^{N} M_i\right)$-position indices

$$J = \left\{j_k\right\}_{k=1}^{\sum_{i=1}^{N} M_i} \text{ by } j_k \in \left\{\overline{1, S+1}\right\} \quad \forall\, k = \overline{1, \sum_{i=1}^{N} M_i} \tag{8}$$

determine the matrix element

$$p_J^{\langle n \rangle} = K_n\left(\mathbf{X}\right) \text{ by } x_{rm} = \frac{j_k - 1}{S}$$

$$\text{at } k = m + \sum_{i=1}^{r-1} M_i \quad \forall\, m = \overline{1, M_r} \text{ and } \forall\, r = \overline{1, N}. \tag{9}$$

Understandably, it is very inconvenient to operate on such multidimensional matrices, representing the players' payoff values. Another inconvenience is computational retardation due to supplementary dimensions. For instance, 1000 operations of summing and extracting mean that minimal and maximal elements of $50 \times 50 \times 50 \times 50$-matrix in Matlab take about 303 seconds, whereas the same takes about 295 seconds over the matrix reshaped into 2500×2500-matrix. Therefore matrices $\left\{\mathbf{P}_n\right\}_{n=1}^{N}$ should be reshaped to reduce the number of their dimensions ultimately. The minimal number of dimensions, apparently, is the number of players.

Before getting started, note that the subset of indices $\left\{j_k\right\}_{k=1+\sum_{i=1}^{n-1} M_i}^{\sum_{i=1}^{n} M_i} \subset J$ corresponds to a pure strategy of the *n*th player, $n = \overline{1, N}$. Then F-matrix $\mathbf{P}_n = \left[p_J^{\langle n \rangle}\right]_{\mathsf{F}}$ is reshaped into $\bigtimes_{r=1}^{N} \left(S+1\right)^{M_r}$-matrix

$$\mathbf{G}_n = \left[g_L^{\langle n \rangle}\right]_{\mathsf{L}} \tag{10}$$

of elements $g_L^{\langle n \rangle} = p_J^{\langle n \rangle}$, whose indices within the format $\mathsf{L} = \bigtimes_{r=1}^{N} \left(S+1\right)^{M_r}$ are in the set

$$L = \left\{u_r\right\}_{r=1}^{N} \text{ by } u_r = \sum_{m=1}^{M_r} \left(S+1\right)^{m-1} \cdot \left(j_k - \mathrm{sign}\left(m-1\right)\right)$$

$$\text{at } k = \sum_{i=1}^{r} M_i - m + 1 \text{ and } \forall\, r = \overline{1, N}. \tag{11}$$

Henceforward, instead of the infinite noncooperative game (1) on unit hypercube (4) here is its approximation as the finite noncooperative game

$$\left\langle \left\{\left\{z_{u_n}^{\langle \mathbf{X}_n \rangle}\left(S\right)\right\}_{u_n=1}^{\left(S+1\right)^{M_n}}\right\}_{n=1}^{N}, \left\{\mathbf{G}_n\right\}_{n=1}^{N} \right\rangle, \tag{12}$$

where the *n*th player's pure strategy $z_{u_n}^{\langle \mathbf{X}_n \rangle}\left(S\right)$ corresponds to its strategy $\mathbf{X}_n$ in the initial game (1), whose components are

$$\left\{x_{nm} = \frac{j_k - 1}{S}\right\}_{m=1}^{M_n} \text{ at } k = m + \sum_{i=1}^{n-1} M_i \text{ by } n = \overline{1, N}. \tag{13}$$

For the *n*th player, the transition from the matrix $\mathbf{P}_n = \left[p_J^{\langle n \rangle}\right]_{\mathsf{F}}$ to the matrix (10), implying the mapping of the indices set (8) into the indices set (11), is reversible. This is maintained by that the subset of indices $\left\{j_k\right\}_{k=1+\sum_{i=1}^{n-1} M_i}^{\sum_{i=1}^{n} M_i} \subset J$, corresponding to the pure strategy $z_{u_n}^{\langle \mathbf{X}_n \rangle}\left(S\right)$ in the game (12), is found by the index u_n:

$$j_k = \eta\left(u_n, S+1\right) + \left(S+1\right)\left(1 - \mathrm{sign}\left[\eta\left(u_n, S+1\right)\right]\right)$$

$$\text{at } k = \sum_{i=1}^{n} M_i,$$

$$j_{k-m} = 1 + \eta\left(\frac{u_n - j_k - \sum_{q=1}^{m-1} \left(S+1\right)^q \left(j_{k-q} - 1\right)}{\left(S+1\right)^m}, S+1\right)$$

$$\forall\, m = \overline{1, M_n - 1} \tag{14}$$

by the function $\eta\left(y, b\right)$ returning the fractional part (remainder after division) of the ratio $\frac{y}{b}$. Then the set (8) is indeed given back, whence the matrix element (9) can be restored.

VI. Consistency of NE-Strategy Support, Approximating the Unknown Genuine NE-Strategy

May the set

$$\left\{\left\{p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{X}_n \rangle}\left(S\right)\right)\right\}_{u_n=1}^{\left(S+1\right)^{M_n}}\right\}_{n=1}^{N} \tag{15}$$

be the game (12) solution, in which $p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{X}_n \rangle}\left(S\right)\right)$ is the probability of applying the pure strategy $z_{u_n}^{\langle \mathbf{x}_n \rangle}(S)$ in an NE-strategy of the nth player

$$\left\{ p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{x}_n \rangle}(S) \right) \right\}_{u_n=1}^{(S+1)^{M_n}} . \quad (16)$$

And may the support of the nth player's NE-strategy (16) be the set

$$\left\{ z_{u_n^{\langle q \rangle}(S)}^{\langle \mathbf{x}_n \rangle}(S) \right\}_{q=1}^{Q_n^*(S)} , \ \left\{ u_n^{\langle q \rangle}(S) \right\}_{q=1}^{Q_n^*(S)} \subset \left\{ \overline{1, (S+1)^{M_n}} \right\} , \quad (17)$$

having its cardinality $Q_n^*(S) \leqslant (S+1)^{M_n}$ by the number S of intervals between the selected points in each of dimensions of hypercube U_n. Then the nth player's NE-strategy (16) has the property:

$$p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{x}_n \rangle}(S) \right) > 0 \quad \forall\, u_n \in \left\{ u_n^{\langle q \rangle}(S) \right\}_{q=1}^{Q_n^*(S)} \quad (18)$$

and

$$p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{x}_n \rangle}(S) \right) = 0 \quad \forall\, u_n \notin \left\{ u_n^{\langle q \rangle}(S) \right\}_{q=1}^{Q_n^*(S)} . \quad (19)$$

The task now is to see how badly the approximate solution will change if the number of intervals between the selected points in each of dimensions of hypercubes $\{U_n\}_{n=1}^{N}$ is increased minimally. For that there must be compared not just approximate solutions (15) and

$$\left\{ \left\{ p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{x}_n \rangle}(S+1) \right) \right\}_{u_n=1}^{(S+2)^{M_n}} \right\}_{n=1}^{N} , \quad (20)$$

but also the players' payoffs. And conception of the Helly metric [6, 10, 11] couldn't be applied here inasmuch as the couple of each player's NE-strategies from solutions (15) and (20) is compiled from diverse games.

In this connection denote by $v_{NE}^{\langle n \rangle}(S)$ the nth player's payoff, being taken in the factual NE-situation (15):

$$v_{NE}^{\langle n \rangle}(S) = \sum_{\substack{L=\{u_l\}_{l=1}^{N} \\ u_l=\overline{1,(S+1)^{M_l}}}} \left(g_L^{\langle n \rangle} \cdot \prod_{i=1}^{N} p_{NE}^{\langle i \rangle}\left(z_{u_i}^{\langle \mathbf{x}_i \rangle}(S) \right) \right) =$$

$$= \sum_{\substack{L_*(S)=\left\{u_l^{\langle q \rangle}(S)\right\}_{l=1}^{N} \\ q=\overline{1,Q_n^*(S)}}} \left(g_{L_*(S)}^{\langle n \rangle} \cdot \prod_{i=1}^{N} p_{NE}^{\langle i \rangle}\left(z_{u_i^{\langle q \rangle}(S)}^{\langle \mathbf{x}_i \rangle}(S) \right) \right), \quad n=\overline{1, N} . \quad (21)$$

So, N players take their payoffs $\left\{ v_{NE}^{\langle n \rangle}(S) \right\}_{n=1}^{N}$ in NE-situation (15), and they take payoffs $\left\{ v_{NE}^{\langle n \rangle}(S+1) \right\}_{n=1}^{N}$ in NE-situation (20) for the minimally increased sampling number (the minimally decreased sampling step). Apparently, there can be selected such a sampling step, for which at least $\exists\, n_0 \in \left\{ \overline{1, N} \right\}$ such that payoffs $v_{NE}^{\langle n_0 \rangle}(S)$ and $v_{NE}^{\langle n_0 \rangle}(S+1)$ will be significantly different. Another difference is that NE-situation (15) in the game (12) will have configuration, being hardly comparable to the corresponding configuration of NE-situation in the game (1) or to NE-situation (20) for the minimally decreased sampling step. So, for initial acceptance of the game (12) solution as an approximate solution of the game (1) there are the following exigencies:

1) a sufficient closeness of the players' NE-situations, being found by nearest neighbor numbers of intervals between the selected points in each of dimensions of every hypercube from $\{U_n\}_{n=1}^{N}$;

2) a sufficient closeness of the players' payoffs, being taken in these NE-situations.

However, what is the rate of "sufficient closeness" for those ones? Obviously, it is unknown as well as the players' payoffs are. The one that remains there is a relative closeness, meaning that the attribute value (the player's payoff, the player's NE-strategy, etc.) differentiates less as the number S increases (growing more "stable"). In the payoff case, that relative closeness of the players' payoffs is this

$$\left| v_{NE}^{\langle n \rangle}(S) - v_{NE}^{\langle n \rangle}(S+1) \right| \leqslant \left| v_{NE}^{\langle n \rangle}(S-1) - v_{NE}^{\langle n \rangle}(S) \right|$$
$$\forall\, n = \overline{1, N} . \quad (22)$$

The sufficient closeness in the case of NE-situations, giving payoffs for players, needs consideration of the player's NE-strategies supports as hypersurfaces. Let for the nth player there be a piecewise linear hypersurface $h_n(u_n, S)$, vertices of which are in points

$$\left\{ \left\{ \left[\frac{j_k - 1}{S} \right]_{1 \times M_n} \in \mathbb{R}^{M_n} : k = m + \sum_{i=1}^{n-1} M_i ,\ m = \overline{1, M_n} \right\}, \right.$$
$$\left. p_{NE}^{\langle n \rangle}\left(z_{u_n}^{\langle \mathbf{x}_n \rangle}(S) \right) \right\} \quad (23)$$

in the space $\mathbb{R}^{M_n+1}$. The nth player's NE-strategy support (17) scores up the nonzero vertices of the hypersurface $h_n(u_n, S)$ by (18) and (19), wherein

$$\mathbf{X}_n^{\langle q \rangle}(S) = \left[x_{nm}^{\langle q \rangle}(S) \right]_{1 \times M_n} = \left[\frac{j_k^{\langle q \rangle}(S) - 1}{S} \right]_{1 \times M_n} \in U_n$$

$$\text{by} \quad k = m + \sum_{i=1}^{n-1} M_i \quad \text{and} \quad m = \overline{1, M_n} \quad (24)$$

by (13) for $q = \overline{1, Q_n^*(S)}$ and matching the index $u_n^{\langle q \rangle}(S)$ to the point (24) through the expansion (14). Then may the nth player's set $\left\{ \mathbf{X}_n^{\langle q \rangle}(S) \right\}_{q=1}^{Q_n^*(S)}$ of the points (24) be sorted into the set

$$\left\{ \bar{\mathbf{X}}_n^{\langle q \rangle}(S) \right\}_{q=1}^{Q_n^*(S)} = \left\{ \left[\frac{\bar{j}_k^{\langle q \rangle}(S) - 1}{S} \right]_{1 \times M_n} \right\}_{q=1}^{Q_n^*(S)} = \left\{ \mathbf{X}_n^{\langle q \rangle}(S) \right\}_{q=1}^{Q_n^*(S)}$$

$$\text{by}\quad k=m+\sum_{i=1}^{n-1}M_i \quad\text{and}\quad m=\overline{1,\,M_n} \tag{25}$$

so that the value

$$\min_{q_1\in\left\{q+1,\,Q_n^*(S)\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\ m=\overline{1,\,M_n}}\left(\overline{j}_k^{\langle q\rangle}(S)-\overline{j}_k^{\langle q_1\rangle}(S)\right)^2} \tag{26}$$

is reached at $q_1=q+1$ for each $q=\overline{1,\,Q_n^*(S)-1}$ $\forall\, n=\overline{1,\,N}$. With new indices $\left\{\overline{j}_k^{\langle q\rangle}(S)\right\}_{q=1}^{Q_n^*(S)}$ of the sets (25) $\forall\, n=\overline{1,\,N}$ there appears an enhancement in measuring the averaged support density. This is needed because the inequality

$$Q_n^*(S+1)\geqslant Q_n^*(S)\quad\forall\, n=\overline{1,\,N} \tag{27}$$

cannot singly itself express the requirement of that the averaged support density shall not decrease for the minimally decreased sampling step (the minimally increased sampling number).

Definition 1. The solution (15) of the game (12) is called weakly consistent for being the approximate solution of the game (1) if $\forall\, n=\overline{1,\,N}$ the inequalities

$$\max_{q\in\left\{1,\,Q_n^*(S+1)-1\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\ m=\overline{1,\,M_n}}\left(\overline{j}_k^{\langle q\rangle}(S+1)-\overline{j}_k^{\langle q+1\rangle}(S+1)\right)^2}\leqslant$$

$$\leqslant\max_{q\in\left\{1,\,Q_n^*(S)-1\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\ m=\overline{1,\,M_n}}\left(\overline{j}_k^{\langle q\rangle}(S)-\overline{j}_k^{\langle q+1\rangle}(S)\right)^2}, \tag{28}$$

$$\max_{U_n}\left|h_n(u_n,\,S)-h_n(u_n,\,S+1)\right|\leqslant$$

$$\leqslant\max_{U_n}\left|h_n(u_n,\,S-1)-h_n(u_n,\,S)\right| \tag{29}$$

and

$$\left\|h_n(u_n,\,S)-h_n(u_n,\,S+1)\right\|\leqslant$$

$$\leqslant\left\|h_n(u_n,\,S-1)-h_n(u_n,\,S)\right\|\ \text{ in }\ \mathbb{L}_2(U_n) \tag{30}$$

are true along with (22) and (27).

Noting that inequalities (27) and (28) might be stated for the minimally increased sampling step, there is a way to condition the approximate solution harder. This, nonetheless, will underscore the “monotonicity” of NE-strategies in (15) within minimal neighborhood of the sampling step.

Definition 2. The weakly consistent solution (15) of the game (12) is called consistent for being the approximate solution of the game (1) if $\forall\, n=\overline{1,\,N}$ the inequalities

$$Q_n^*(S)\geqslant Q_n^*(S-1) \tag{31}$$

and

$$\max_{q\in\left\{1,\,Q_n^*(S)-1\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\ m=\overline{1,\,M_n}}\left(\overline{j}_k^{\langle q\rangle}(S)-\overline{j}_k^{\langle q+1\rangle}(S)\right)^2}\leqslant$$

$$\leqslant\max_{q\in\left\{1,\,Q_n^*(S-1)-1\right\}}\sqrt{\sum_{k=m+\sum_{i=1}^{n-1}M_i,\ m=\overline{1,\,M_n}}\left(\overline{j}_k^{\langle q\rangle}(S-1)-\overline{j}_k^{\langle q+1\rangle}(S-1)\right)^2} \tag{32}$$

are true.

Surely, (weak) consistency of an NE-strategy support, approximating the unknown genuine NE-strategy, is not sufficient to say that the solution (15) is (weakly) consistent. Speaking strictly, for now there is no proof that the consistent NE-strategy support causes at least the weak consistency of the other NE-strategy support. That is for admission of the solution (15) as an approximate solution of the game (1) there are $5N$ inequalities (22) and (27)–(30) to be checked. And if one wants to handle the game (1) approximation surer there are $7N$ inequalities (22) and (27)–(32) to be checked.

Properly speaking, neither conditions within Definition 1, nor conditions within Definition 2 guarantee the perfection of the game (1) approximation as the game (12) with its (weakly) consistent solution (15). But with (22), (29), (30) there is the solution (15) distinctive property signifying that both the players’ payoffs (21) and the players’ NE-strategies supports as hypersurfaces differentiate less as the number S increases. Growing more “stable”, the volume and the averaged NE-strategies supports’ densities also do not decrease as the number S increases due to (27) and (28). This “non-decreasing” property becomes stronger with (31)–(27) and (32)–(28), strengthening the solution (15) relative independence upon the sampling step within its minimal neighborhood.

VII. Discussion and Conclusive Remarks

The conception of consistency has been contrived for proper approximation of the infinite noncooperative game. Being defined on the unit hypercube (4), this game is isomorphic to games, defined on compact subspaces in $\mathbb{R}^{\sum_{n=1}^{N}M_n}$, wherein the nth player acts within the compact subspace of $\mathbb{R}^{M_n}$ by $n=\overline{1,\,N}$. According to the isomorphism, for solving infinite noncooperative games on compact action spaces there is available the stated approximation way to be applied, allowing to reshape multidimensional matrices of players’ payoff values by (11) to the minimal number of dimensions, being equal to the number of players. Due to narrowing the dimensionality, this anyhow shortens the computation period. And computing the factual solution stays for finite noncooperative game solvers [1, 10, 38, 43, 44].

Before approximating, the weak consistency ought to be checked first. The check consecution starts with checking the inequalities (27), where two games are solved towards (15) and (20). Then goes subconsecution of checking the inequalities (22), (29) and (30) $\forall\, n=\overline{1,\,N}$, needing three games to be solved, towards (15), (20) and

$$\left\{\left\{p_{NE}^{\langle n\rangle}\left(z_{u_n}^{\langle\mathbf{x}_n\rangle}(S-1)\right)\right\}_{u_n=1}^{S^{M_n}}\right\}_{n=1}^{N}. \tag{33}$$

Finally, the inequalities (28) $\forall\, n=\overline{1,\, N}$ are checked, needing more computational resources for sorting sets $\left\{\left\{\mathbf{X}_n^{\langle q\rangle}\left(S\right)\right\}_{q=1}^{Q_n^*(S)}\right\}_{n=1}^{N}$ into sets $\left\{\left\{\bar{\mathbf{X}}_n^{\langle q\rangle}\left(S\right)\right\}_{q=1}^{Q_n^*(S)}\right\}_{n=1}^{N}$ although using the previously solved two games with (15) and (20). When the weakly consistent solution (15) of the game (12) is checked for consistency, there are used solutions (15) and (33) once again, and then N inequalities (31) are checked first, whereupon come those N inequalities (32). Namely the stated consecutions are preferable, because the easiest requirements are checked before the more complicated ones in order to prevent needless huge computations (manipulations) over non-consistent solutions, being exposed after easier comparisons like (27) or (22).

The main lack of the approximation is that there is neither proved limits

$$\lim_{S\to\infty} v_{NE}^{\langle n\rangle}\left(S\right) \quad \forall\, n=\overline{1,\, N}$$

existence and their convergence to the genuine players' payoffs in NE-situation approximated by NE-situation (15), nor proved limits

$$\lim_{S\to\infty} h_n\left(u_n,\, S\right) \quad \forall\, n=\overline{1,\, N}$$

existence and their convergence to the hypersurfaces from the genuine NE-situation approximated by NE-situation (15). But anyway, the represented method of converting the infinite noncooperative game on unit hypercube into the finite game lets to have an NE-solution to the conflict object, even when the game (1) is solved in ε-equilibrium situations or doesn't have solution at all. Besides, the approximate solution (15), where every player has the finite NE-strategy support, is practiced more freely with discrete variates [45], [46] unlike practicing on continuous variates [47], [48].

The investigation of approximating infinite noncooperative games could be brought forward if the set in (5) for *n*th player was formed with a specific number of intervals S_n between the selected points in each of hypercube U_n dimensions. Further to this, *n*th player could take a specific number of intervals S_{nm} between the selected points in *m*th dimension of one's pure strategy (2). This might be useful inasmuch as multidimensional matrices of players' payoff values in the hypercubic lattice form are preferred to other forms [10]. However, the main issue is to make problems of the type (26) effectively computable for accelerating the consistency checks in (28) and (32).

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